

Do four problems of your choice. Cite any other students or resources which you used.

Exercise 1. For each of the following regions, give explicit isomorphisms with $\mathbb{D}$ or $\mathbb{H}$.

- (a) The infinite band $B = \{z \in \mathbb{C} \mid 0 < \operatorname{Im} z < \pi\}$.
- (b) The half-infinite band $\{z \in \mathbb{C} \mid 0 < \operatorname{Im} z < \pi, \operatorname{Re} z > 0\}$.
- (c) The taco silhouette $\{z \in \mathbb{C} \mid \operatorname{Im} z > 0, |z| < 1\}$.
- (d) The “Lucio Fontana” $\hat{\mathbb{C}} - [0, 1]$.

Exercise 2.

- (a) Prove that the action of $\operatorname{PSL}(2, \mathbb{C})$ on $\hat{\mathbb{C}}$ is *simply 3-transitive*: there is a unique Möbius transformation sending three specified points to three other specified points in $\hat{\mathbb{C}}$. Deduce the dimension of $\operatorname{PSL}(2, \mathbb{C})$ as a complex manifold.
- (b) Prove that $\operatorname{PSL}(2, \mathbb{C})$ is generated by the affine maps $z \mapsto az + b$ along with the inversion $z \mapsto 1/z$.
- (c) Prove that the cross-ratio of four points

$$[z_1, z_2, z_3, z_4] = \frac{(z_3 - z_1)(z_4 - z_2)}{(z_2 - z_1)(z_4 - z_3)}$$

is preserved under Möbius transformations.

Exercise 3 (Hyperbolic geometry potpourri).

- (a) (Optional) Write down the hyperbolic metric on $\mathbb{D}$.
- (b) Prove that the action of $\operatorname{PSL}(2, \mathbb{R})$ is simply transitive on the unit tangent bundle to $\mathbb{H}$: there is a unique isometry which sends a given point z_1 to another point z_2 in $\mathbb{H} \cong \mathbb{D}$ and a specified tangent vector of unit length at z_1 to a tangent vector of unit length at z_2 . Deduce the dimension of $\operatorname{PSL}(2, \mathbb{R})$ as a real manifold.
- (c) Prove that $\operatorname{PSL}(2, \mathbb{R})$ sends three specified points on $\partial\mathbb{H}$ to three other points if and only if the triples have the same cyclic order.
- (d) The *translation length* of a hyperbolic isometry $A \in \operatorname{PSL}(2, \mathbb{R})$ is defined to be

$$d(A) = \inf\{d_{\mathbb{H}}(z, A \cdot z) \mid z \in \mathbb{H}\}.$$

Prove that $d(A) = 0$ and the infimum is attained iff A is elliptic, $d(A) = 0$ and the infimum is not attained iff A is parabolic, and $d(A) > 0$ iff A is hyperbolic.

In the last case, prove that there is a unique geodesic on which $d(A)$ is attained and A acts by translation. Relate $d(A)$ to the trace of A .

- (e) Prove that a circle in the hyperbolic metric is a Euclidean circle in $\mathbb{D}$ or $\mathbb{H}$, but with a possible different center.

- (f) Prove that two hyperbolic triangles are congruent if and only if they have the same angles.
Hint: Use Euclidean geometry.
- (g) (Optional) Using calculus, calculate the area of the *ideal triangle* with vertices at $0, 1, \infty \in \mathbb{H}$ is π .
- (h) Assume the previous part. Let $A(\theta)$ be the area of a triangle with angles $(0, 0, \pi - \theta)$. Prove that $A(\theta_1 + \theta_2) = A(\theta_1) + A(\theta_2)$. (See Figure 1) Deduce that $A(\theta) = \theta$.
Finally prove that the area of a triangle with angles (α, β, γ) is $\pi - (\alpha + \beta + \gamma)$.

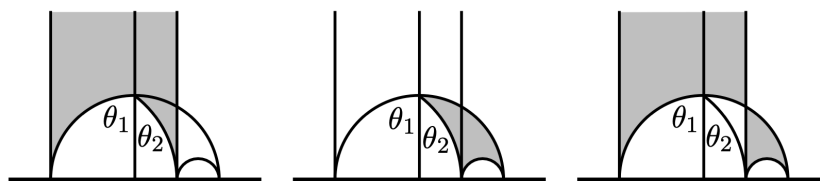


Figure 1: Triangles with angles $(0, 0, \theta)$.

- (i) (Optional) Recall (or prove to your own satisfaction) that one gets an orientable surface of genus 2 by gluing sides of an octagon according to the pattern specified in Figure 2.

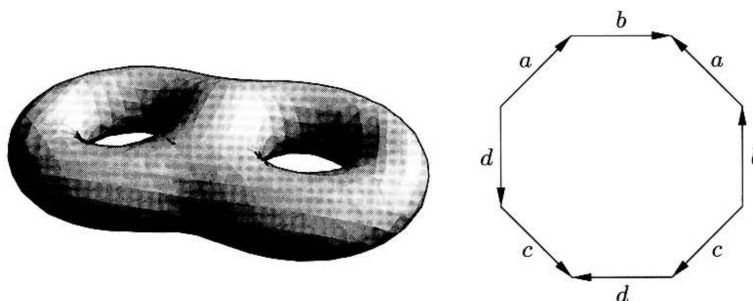


Figure 2: Octagonal model for a genus 2 surface.

A regular octagon does not tile the Euclidean plane because its interior angles are too large. Using Figure 3 as inspiration, prove that there is a regular hyperbolic octagon with internal angles $\pi/4$.

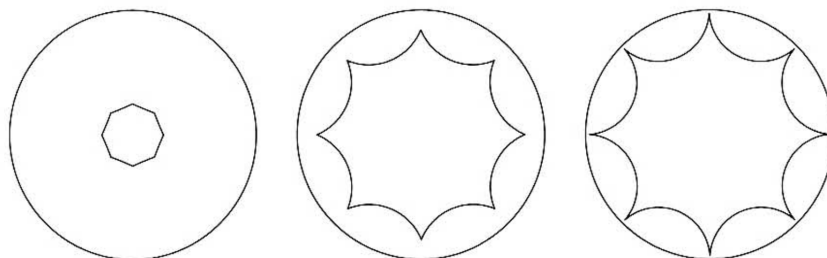


Figure 3: Hyperbolic octagons.

Now show that the side gluings can be realized by isometries, so that the quotient is a Riemann surface / hyperbolic surface. (Hint: Use hyperbolic reflections)

This method generalizes to a construction of Riemann surface structures, or equivalently, hyperbolic structures, on orientable surfaces of all genera $g \geq 1$. It should be noted that they are far from unique (cf. Exercise 5)

- (j) Prove that there is at least one geodesic connecting any two points on a hyperbolic surface.

Exercise 4 (Classification of annuli).

- (a) Prove that $\mathbb{C}^\times \cong \mathbb{C}/\mathbb{Z}$. This is called the “doubly-infinite cylinder”. Optional: Write down the Euclidean metric on $\mathbb{C}^\times$.
- (b) Prove that $\mathbb{D} - \{0\} \cong \mathbb{H}/\langle p \rangle$ for $p \in \text{PSL}(2, \mathbb{R})$ any parabolic. Figure 2 in Milnor is a picture of the hyperbolic geometry on this surface.

Prove that any geodesic either goes straight into the cusp or wraps around finitely many times.

This surface is called the “singly-infinite cylinder” because if we consider $\mathbb{H} \subset \mathbb{C}$ with the induced (incomplete) Euclidean metric, then $\mathbb{H}/\langle z \mapsto z + 1 \rangle$ is a cylinder with one end going off to infinity.

- (c) Let B be the infinite band from Exercise 1 (a). You can assume the result that $B \cong \mathbb{D}$.

Let $0 < M < \infty$. Prove that the following are isomorphic:

$$A_M = \{1 < |z| < e^{2\pi M}\}, \quad B/\langle z \mapsto z + \frac{\pi}{M} \rangle, \quad \mathbb{H}/\langle z \mapsto e^{\pi/M} z \rangle.$$

The number M is called the “modulus”. Prove that $A_M \cong A_{M'}$ iff $M = M'$. (Hint: Lift a putative isomorphism to the universal cover)

With its hyperbolic geometry, A_M looks like the surface in Figure 4.

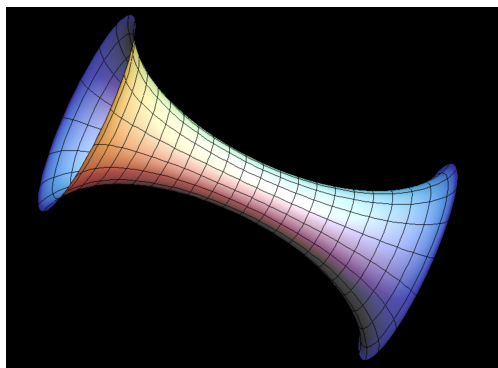


Figure 4: Part of a hyperbolic annulus of finite modulus.

Prove that there is a unique closed geodesic on A_M (the “core curve”) and that it has hyperbolic length π/M . (See leftmost Figure 5)

Prove that a geodesic on A_M is either the core curve, stays on one side of the core curve and wraps around finitely many times, crosses the core curve once and wraps around finitely many times, or stays on one side of the core curve and spirals towards it. (See Figure 5.)

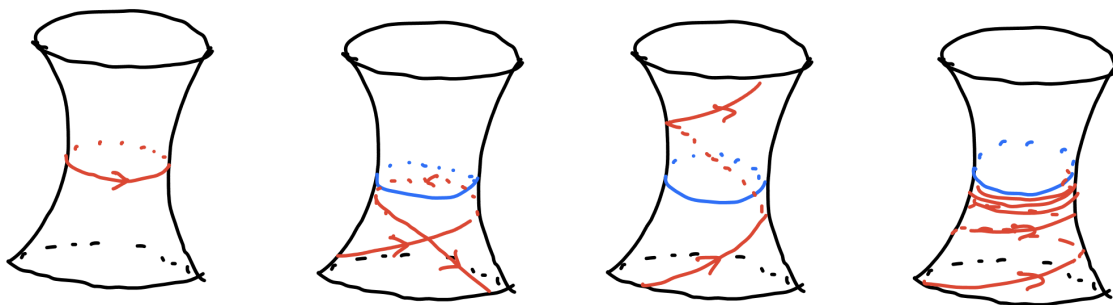


Figure 5: Four behaviors for geodesics on a hyperbolic annulus.

In part (b) we saw that even though $D - \{0\}$ is hyperbolic, it has a description in terms of Euclidean geometry as a half-infinite cylinder. Observe that the second model for A_M as a quotient of a band describes A_M as the interior of a Euclidean cylinder where the ratio of the height to the circumference is M .

Exercise 5 (Classification of elliptic curves).

- (a) By lifting to the universal cover, prove that two elliptic curves $\mathbb{C}/\Lambda_1$ and $\mathbb{C}/\Lambda_2$ are isomorphic iff $\mu \cdot \Lambda_1 = \Lambda_2$ for some $\mu \in \mathbb{C}^\times$.
- (b) Prove that every elliptic curve is isomorphic to $\mathbb{C}/\mathbb{Z}[\tau]$ for some $\tau \in \mathbb{H}$.
- (c) Let $\lambda \in \mathbb{C}^\times$. Give a $\tau \in \mathbb{H}$ so that $\mathbb{C}^\times / \langle z \mapsto \lambda z \rangle$ is isomorphic to $\mathbb{C}/\mathbb{Z}[\tau]$.
- (d) Prove that $\mathbb{C}/\mathbb{Z}[\tau_1] \cong \mathbb{C}/\mathbb{Z}[\tau_2]$, where $\tau_i \in \mathbb{H}$, iff $A \cdot \tau_1 = \tau_2$ for some $A \in \text{SL}(2, \mathbb{Z})$ acting by Möbius transformations.

Conclude that elliptic curves are classified by the points of $\mathbb{H}/\text{SL}(2, \mathbb{Z})$. This is called the “modular curve”.

- (e) Let us see what the modular curve looks like. Consider the region

$$F = \{z \in \mathbb{H} \mid |\text{Re}(z)| \leq 1/2, |z| \geq 1\}$$

and the two matrices $S, T \in \text{SL}(2, \mathbb{Z})$ given by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

First prove that for every $z \in \mathbb{H}$, there is a $A \in \text{SL}(2, \mathbb{Z})$ for which $A \cdot z \in F$.

- (f) Prove if there is an $A \in \text{SL}(2, \mathbb{Z})$ and $z_1, z_2 \in F$ for which $A \cdot z_1 = z_2$, then either $\text{Re}(z_1) = \pm 1/2$ and $z_2 = z_1 \mp 1$ or $|z_1| = 1$ and $z_2 = -1/z_1$. (Hint: Why can one assume that $\text{Im}(z_2) \geq \text{Im}(z_1)$ and hence $|cz_1 + d| \leq 1$? Then take cases for c .)
- (g) Prove that S and T generate $\text{PSL}(2, \mathbb{Z})$. (Hint: Map $A \cdot (2i)$ back into F for $A \in \text{SL}(2, \mathbb{Z})$.)
- (h) What points on F are fixed by S and T ? By the previous part, $\mathbb{H}/\text{SL}(2, \mathbb{Z}) = F/\langle S, T \rangle$. Draw a picture of the modular curve.

Exercise 6 (The doubling map). The doubling map describes the dynamics on the Julia set of $z \mapsto z^2$ and will be very important for our study of the Mandelbrot set.

Let $E : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ be the continuous map $x \mapsto 2x \pmod{1}$.

- (a) Give a cycle of period three for E .
- (b) Characterize which points $x \in \mathbb{R}/\mathbb{Z}$ are periodic under E in terms of number theory. How many have period n ?
- (c) Characterize which points are strictly preperiodic under E in terms of number theory. (Strictly preperiodic means that $E^{n+k}(x) = E^n(x)$ for some n, k but x is not periodic.)
- (d) Prove that for every x , the backwards orbit $\{y \mid E^n(y) = x \text{ for some } n\}$ is dense.
- (e) Prove that there is an x for which the forward orbit $\mathcal{O}(x) = \{x, E(x), E^2(x), \dots\}$ is dense.
- (f) Prove that there is an x for which $\overline{\mathcal{O}(x)}$ is uncountable yet nowhere dense.
- (g) A *portrait* is a collection $\mathcal{A} = \{A_1, \dots, A_k\}$ (where we also write $A_{k+1} = A_1$) of disjoint finite subsets $A_i \subset \mathbb{R}/\mathbb{Z}$ such that
 - (i) E maps A_i bijectively onto A_{i+1} preserving the cyclic order of points,
 - (ii) the A_i are pairwise unlinked: if $i \neq j$, then each subset A_i is contained in one component of $\mathbb{R}/\mathbb{Z} - A_j$.

Give an example of a portrait $\mathcal{A} = \{A_1, A_2, A_3\}$ with $|A_i| = 2$.

Exercise 7. Successfully install the computer program “Mandel” and make some pictures with it. Describe any technical obstacles you had to overcome. (I would like to know how to get it working on my laptop.)

Alternatively you can try the program “FractalStream” for Mac. Some complex dynamicists keep around decrepit old Macbooks to run it but I am told that it’s possible to get it to work on modern Macs. Follow these instructions:

1. Visit this link and use the Download button at the upper left to download FractalStream to your Mac. It should be put into the Downloads folder in your home directory. If it is downloaded somewhere else, please move it to your Downloads folder. Do not unzip it yet!
2. Open Terminal and execute these commands:


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cd ~/Downloads
xattr -c Fractalstream.app.zip
unzip FractalStream.app.zip
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 (The unzip command will ask for a password, enter “FractalStream” without the quotes).
3. You should then be able to navigate to the app in Finder and double-click it, and it should open.

You will only have to do this the first time you open it, after that you should be able to double-click the app to open it as per usual.